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Self-intersection number of BPS junctions in backgrounds of three and seven-branes

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Abstract: In a recent paper De Wolfe et al. have shown how to use the self-intersection number of junctions to constrain the BPS spectrum of $\mathcal{N} = 2$, $D = 4$ theories with ADE flavor symmetry arising on a single D3-brane probe in a 7-brane background. Motivated by the existence of more general $\mathcal{N} = 2$, $D = 4$ theories arising on the worldvolume of multiple D3-brane probes we show how to compute the self-intersection number of junctions in the presence of 7-branes and multiple D3-branes.

Keywords: F-Theory, D-branes, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

String junctions [1] play an important role in understanding non-perturbative aspects of type IIB superstring theory. They are required to explain gauge symmetry enhancement when non-local 7-branes coincide [2]. String junctions are constructed from (p, q) strings [3] and are non-perturbative objects.

Their role in determining the BPS spectrum of $\mathcal{N} = 2$ theories in four dimensions has been studied in several papers [4, 5, 6]. The BPS states of the $\mathcal{N} = 2$, $D = 4$ SYM are realized as BPS string junctions connecting a D3-brane to 7-branes. In this probe picture the string junctions representing the BPS states are characterized by the invariant charges associated with the 7-branes [7]. String junctions in the presence of non-local 7-branes have interesting group theoretical properties [7] which can be used together with the bound on the self-intersection number of BPS junctions [5] to constrain the spectrum of $\mathcal{N} = 2$ $D = 4$ SYM with ADE flavor symmetry. For $N_f \leq 4$ flavors all the states allowed by the constraint are realized as BPS states [5, 6]. Three string junctions have also been used to construct 1/4 BPS states in the $\mathrm{SU}(N)$ $\mathcal{N} = 4$, $D = 4$ SYM theory [8, 9]. The large N limit of $\mathcal{N} = 2$ theories arising on the worldvolume of N D3-branes in the 7-brane background was studied in [10].

In this paper we will show how to calculate the self-intersection number of junctions in a 7-brane background in the presence of multiple D3-branes. Eqs. (5.3) and (5.4) are the main results of this paper, they can be used to constrain the BPS spectrum of $\mathcal{N} = 2$, $D = 4$ theory with ADE flavor symmetry arising on multiple D3-branes [11]. The results of such a study will be given elsewhere.

A junction of type IIB string theory compactified on a manifold $B \times S^1$ is a two dimensional surface in M-Theory compactified on a hyper-Kähler manifold X which is an elliptic fibration over the base manifold B. The Kähler class of the elliptic fiber is related to the radius R of S^1 and is given by l_p^3/R, where l_p is the eleven-dimensional Planck length. If there are no 7-branes the fibration is trivial and the manifold X is just the product $B \times T^2$. In the presence of 7-branes the fibration is non-trivial and the location of the 7-branes on the base B determines the location of singular fibers [12]. Type IIB string theory on B is obtained from M-Theory on X by taking the Kähler class of the elliptic fiber to zero.

BPS junctions on the base B are made of strings lying along geodesics. These are obtained by wrapping M2-branes around holomorphic curves of X [13]. A (p,q) string corresponds to an M2-brane wrapped around a holomorphic curve which is a product of a (p,q) cycle of the elliptic fiber and a geodesic in the base representing the position of the string in the $R \to \infty$ limit.

With every junction $\mathbf{J}$ of the type IIB string theory one can associate a two dimensional surface J. The correspondence is such that in the limit $R \to \infty$ the surface J goes to junction $\mathbf{J}$ on the base. This correspondence allows us to associate with every pair of junctions $\mathbf{J}$ and $\mathbf{J}'$ a geometrical quantity, the intersection number $^{\#}(J \cdot J')$,

$$(\mathbf{J}, \mathbf{J}') \equiv {}^{\#}(J \cdot J') \,. \tag{1.1}$$

Since the self-intersection number of a holomorphic curve $J \subset X$ of genus g with b boundary components is given by [14]

$${}^{\#}(J \cdot J) = 2g - 2 + b \,, \tag{1.2}$$

BPS junctions satisfy a constraint, $(\mathbf{J}, \mathbf{J}) \geq -2$.

The self-intersection number of a junction can be calculated in terms of (p,q) charges of the strings forming the junction therefore (1.2) allows us to determine the topology of the curve corresponding to the BPS junction in terms of the charges. We illustrate with a detailed argument how the self-intersection number of a three string junction is related to an $\mathrm{SL}(2,\mathbb{Z})$ invariant number associated with the junction [7, 9]. This $\mathrm{SL}(2,\mathbb{Z})$ invariant plays an important role in characterizing BPS junctions but lacks a direct geometrical interpretation in type IIB string theory.

2. Self-intersection number and the $\mathrm{SL}(2,\mathbb{Z})$ invariant

In this section we will relate the invariants of the junctions with the intersection numbers of the corresponding curves in M-theory. The tension $T_{p,q}$ of a (p,q) string is related to the length of the 1-cycle of the torus on which the M2 brane is wrapped. Since, in the Einstein units,

$$T_{p,q} = \frac{1}{\sqrt{\tau_2}} |p - q\tau| \,, \tag{2.1}$$

therefore a (p, q) string corresponds to the 1-cycle $C_{p,q} = p\alpha - q\beta$. Where α and β are the canonical basis homology cycles of the torus satisfying $^{\#}(\alpha \cdot \beta) = 1$.

With a junction $\mathbf{J}$ in IIB we associate a 2-cycle J in M-theory such that in the limit $R \to \infty$ J goes to $\mathbf{J}$. We associate with two junctions $\mathbf{J}$ and $\mathbf{J}'$ an $\mathrm{SL}(2, \mathbb{Z})$ invariant number $(\mathbf{J}, \mathbf{J}')$. We define this number to be the intersection number of the corresponding curves in M-theory [6, 7, 5],

$$(\mathbf{J}, \mathbf{J}') = {}^{\#}(J \cdot J')\,. \tag{2.2}$$

Consider the string junction $\mathbf{J}$ shown in fig. 1a. The self-intersection number gets contribution from only one point where a (p_1, q_1) string crosses an (p_2, q_2) string as shown in fig. 1b. Near the intersection point the corresponding two dimensional surfaces representing the (p_1, q_1) and the (p_2, q_2) strings in M-theory have product structure, $l_i \times C_{p_i,q_i}$, where l_i is the one dimensional curve on the base.

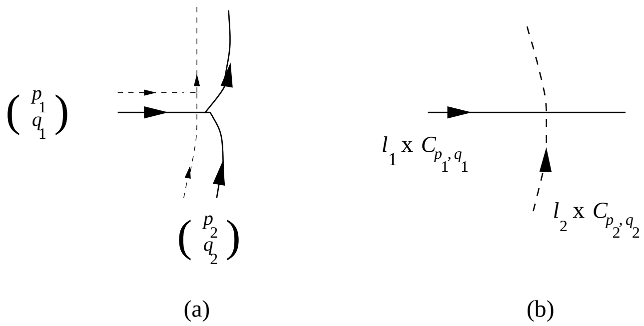

Figure 1: (a) Self-intersection number is calculated by deforming the junction as shown. (b) Near the intersection point the curves have product structure.

Then by standard manipulation [15]

$$\begin{aligned}(\mathbf{J}, \mathbf{J}) = {}^{\#}(J \cdot J) &= {}^{\#}((l_1 \times C_{p_1,q_1}) \cdot (l_2 \times C_{p_2,q_2})) \\ &= -{}^{\#}(l_1 \cdot l_2)\, {}^{\#}(C_{p_1,q_1} \cdot C_{p_2,q_2}).\end{aligned} \tag{2.3}$$

To calculate the intersection number for these 1-cycles we need an orientation on the base and on the fiber. The orientation on the base and the fiber comes from the complex structure of X. Since X is a hyper-Kähler manifold there is a family of possible complex structures compatible with the metric. However we are interested in the complex structures I_+ and I_- in which the fiber is holomorphic. The base is holomorphic in the complex structure I_+ and $-I_-$. Since the fiber is holomorphic therefore $^{\#}(\alpha \cdot \beta) = 1$ and

$${}^{\#}(C_{(p_1,q_1)} \cdot C_{(p_2,q_2)}) = - \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}. \tag{2.4}$$

The intersection number $^{\#}(l_1 \cdot l_2)$ depends on the orientation of the base therefore

$${}^{\#}(l_1 \cdot l_2) = \pm 1, \text{ in the complex structure } I_{\pm}. \tag{2.5}$$

Thus back in (2.3)

$$(\mathbf{J}, \mathbf{J}) = \pm \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}, \text{ in the complex structure } I_{\pm}. \tag{2.6}$$

Thus we see that if J is holomorphic in I_+, $(p_1 q_2 - q_1 p_2) > 0$. In this case a BPS (p,q) string is oriented in the direction $p - q\bar{\tau}$. If J is holomorphic in the complex structure I_- then $(p_1 q_2 - q_1 p_2) < 0$. In this case BPS (p,q) string is oriented in the direction $p - q\tau$.

If there are 7-branes on B the monodromy $K_{p,q}$ around a $[p,q]$ 7-brane is the same as the monodromy around an elliptic fiber whose $p\alpha - q\beta$ cycle is pinched. If τ is a holomorphic function of z (the holomorphic coordinate on the base in the complex structure I_+) then going counterclockwise around the $[p,q]$ 7-brane $\tau(z) \to K_{p,q}\tau(z)$. If τ had been function of $\bar{z}$ then it would have been transformed by $K_{p,q}^{-1}$. Thus the base is holomorphic in the complex structure I_+ when there are 7-branes on it.

In summary, *if* **J** *is a BPS string junction then the* $\mathrm{SL}(2,\mathbb{Z})$ *invariant associated with each junction point is of same sign. If it is positive (negative) for every junction point then the curve corresponding to* **J** *is holomorphic in the complex structure* $I_+(-I_-)$. Thus if in a string web the invariants associated with two junction points are of opposite sign then that string web cannot be BPS. *In the presence of 7-branes therefore the invariant associated with each junction point must be positive for a BPS string junction.*

3. BPS junctions on 3-branes

Consider IIB on $B \times S^1$ with only D3-branes present. Since there are no 7-branes there are no singular fibers and $X = B \times T^2$ is the compactification manifold of M-theory. Consider n D3-branes on B and a BPS string junction **J** between them such that the strings are either all incoming or outgoing so that $\sum_{i=1}^n p_i = \sum_{i=1}^n q_i = 0$. We will associate with this junction a polygon $P(\mathbf{J})$ [12] in the lattice Γ, which is dual to the lattice of string junctions, defined as

$$\Gamma = \{(p,q) = -i(p - q\bar{\tau}) \,|\, p,q \in \mathbb{Z}\}\,. \tag{3.1}$$

This polygon $P(\mathbf{J})$ has vertices $\{(\sum_{i=1}^k p_i, \sum_{i=1}^k q_i) | k = 1,\ldots,n\}$. Since the self-intersection number does not get any contributions from the boundaries we can send the D3-branes to infinity and consider the resulting infinite string junction $\mathbf{J}'$ [16]. Therefore the polygon $P(\mathbf{J}')$ is the same as $P(\mathbf{J})$ and $^\#(J' \cdot J') = {}^\#(J \cdot J)$.

From toric geometry we know the relationship between the area, $A(P(\mathbf{J}))$, of the polygon $P(\mathbf{J})$ and the self-intersection number of holomorphic curve J' [17]:

$$^\#(J' \cdot J') = 2A(P(\mathbf{J})) = 2g - 2 + b\,, \tag{3.2}$$

where a cell of the lattice Γ is declared to have unit area. We define

$$d_i = \mathcal{GCD}(p_i, q_i)\,, \tag{3.3}$$

where $\mathcal{GCD}(p,q) \geq 1$ is the greatest common divisor of p_i and q_i. The number of boundary components b is given by [5]

$$b = \sum_{i=1}^{n} d_i = l(P(\mathbf{J}))\,, \tag{3.4}$$

where $l(P(\mathbf{J}))$ is the number of lattice points on the perimeter of $P(\mathbf{J})$. The number of integral points $\#(P(\mathbf{J}))$ inside the polygon $P(\mathbf{J})$ is given by Pick's formula [17]

$$\#(P(\mathbf{J})) = A(P(\mathbf{J})) - \frac{1}{2} l(P(\mathbf{J})) + 1\,. \tag{3.5}$$

Thus from (3.2) and (3.5)

$$g = \frac{1}{2}^{\#}(J \cdot J) - \frac{1}{2} b + 1 = A(P(\mathbf{J})) - \frac{1}{2} l(P(\mathbf{J})) + 1 = \#(P(\mathbf{J}))\,. \tag{3.6}$$

In terms of the string charges (p_i, q_i) we get

$$^{\#}(J \cdot J) = 2A(P(\mathbf{J})) = \left| \sum_{1 \leq i < j \leq n} \begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix} \right| \tag{3.7}$$

$$g = \#(P(\mathbf{J})) = \frac{1}{2}^{\#}(J \cdot J) - \frac{1}{2} \sum_{i=1}^{n} d_i + 1\,. \tag{3.8}$$

If the junction $\mathbf{J}$ is BPS and all the strings are oriented in the direction $p - q\bar{\tau}$ then each term in the sum (3.7) is positive and the area of $P(\mathbf{J})$ is the sum of these terms. If the strings are oriented in the direction $p - q\tau$ then each term in the sum is negative and the area is minus the sum of these negative terms. Thus we need to take the absolute value of the sum in (3.7).

4. BPS junctions on 7-branes

7-branes of type IIB are the singular fibers of a hyper-Kähler manifold X in the lift to M-theory. 2-cycles with support on the 7-branes do not have boundaries. Therefore in the presence of 7-branes (3.6) fails and the number of interior points in the polygon $P(\mathbf{J})$ is not equal to the genus of the corresponding holomorphic curve J. The self-intersection number of a genus g holomorphic curve $J \subset X$ without boundary is (1.2)

$$^{\#}(J \cdot J) = 2g - 2\,. \tag{4.1}$$

In fact in a homology class $J \in H^2(X)$ of self-intersection number greater than or equal to minus two the minimal area surface is a holomorphic curve of genus $^{\#}(J \cdot J) + 2/2$ [18]. This implies that a junction corresponding to a surface of self-intersection number greater than or equal to minus two will always have a BPS representative. This BPS representative corresponds the curve of least genus and minimal area in this homology class. It will be unique if the self-intersection number is minus two [18]. This has been explained in the type IIB viewpoint in [19].

Consider a junction $\mathbf{J}$ which ends on 7-branes. The strings ending on the 7-branes cannot be deformed away from the 7-branes and therefore also contribute to the self-intersection number. This additional contribution changes the relation between the number of interior points in $P(\mathbf{J})$ and the genus of J. If a (p_i, q_i) string ends on a $[p_i, q_i]$ 7-brane then the contribution to the self-intersection due to the presence of the 7-brane is $-d_i^2$. (A (p_i, q_i) string, with p_i and q_i relatively prime, ending on a 7-brane is like a half sphere with a marked point. A general (p_i, q_i) string is a sphere wrapped d_i times around itself. The contribution to the self-intersection number from one of the fixed points of the sphere is $\frac{1}{2}{}^{\#}(\Sigma \cdot \Sigma) = \frac{1}{2}(-2)$, where Σ is the two sphere. With n-prongs we get a contribution equal to one half the self-intersection of a sphere wrapped n-times around it self which is equal to $\frac{1}{2}{}^{\#}(n\Sigma \cdot n\Sigma) = -n^2 = -d_i^2$ [20].)[1]

Since the self-intersection number does not change under a continuous deformation of J, we deform the junction $\mathbf{J}$ such that the strings do not cross the branch cuts associated with the 7-branes. The total self-intersection number is the sum of self-intersection numbers from the string endpoints on the 7-branes and the junction points. The contribution to the total self-intersection number from the junction points is equal to the self-intersection number of an infinite junction with same charges. Thus

$$
{}^{\#}(J \cdot J) = 2A(P(\mathbf{J})) - \sum_{i=1}^{n} d_i^2 \,. \tag{4.2}
$$

The last term is the contribution from the end points of the strings on the 7-branes. In terms of string charges (p_i, q_i) we get

$$
{}^{\#}(J \cdot J) = \sum_{1 \le i < j \le n} \begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix} - \sum_{i=1}^{n} d_i^2 \,. \tag{4.3}
$$

This is in agreement with [6, 7]. Using (4.1) we get

$$
g = \#(P(\mathbf{J})) + \frac{1}{2} \sum_{i=1}^{n} \{d_i - d_i^2\} = \frac{1}{2}{}^{\#}(J \cdot J) + 1 \,. \tag{4.4}
$$

Thus we see that holomorphic curves which correspond to junctions with ends on the 7-branes with non-relatively prime charges have lower genus than a junction with same charges ending on D3-branes (see (3.8)).

[1] Since the junctions we are considering have no asymptotic charge therfore the quadratic form, defined in [7], $f(p, q) \equiv 0$ in this case.

5. BPS junctions on 3-branes and 7-branes

Consider a BPS junction $\mathbf{J}$ in the presence of n 7-branes and m D3-branes. The corresponding holomorphic curve J has b boundary components (from strings ending on D3-branes) and n marked points (from strings ending on 7-branes).

We choose a presentation $\mathbf{J}'$ of the junction $\mathbf{J}$ in which strings do not cross the branch cuts associated with the 7-branes and the D3-branes are grouped together far from the 7-branes. This is achieved by continuous deformation of the junction which leaves the self-intersection numbers invariant. Let $\{(p_i, q_i), i = 1, \ldots, n\}$ and $\{(p_i, q_i), i = n+1, \ldots, n+m\}$ be the charges of the strings forming the junction $\mathbf{J}'$ and ending on 7-branes and D3-branes respectively. We define $P(\mathbf{J}')$ as the polygon with vertices $\{(\sum_{i=1}^{k} p_i, \sum_{i=1}^{k} q_i) | k = 1, \ldots, n+m\}$. Then

$$^{\#}(J \cdot J) =^{\#} (J' \cdot J') = 2g - 2 + b = 2A(P(\mathbf{J}')) - \sum_{i=1}^{n} d_i^2 \,. \tag{5.1}$$

The last term is the contribution from the strings ending on 7-branes. The number of boundary components b is given by

$$b = \sum_{i=n+1}^{n+m} d_i \,, \qquad \text{and} \qquad l(P(\mathbf{J}')) - b = \sum_{i=1}^{n} d_i \,. \tag{5.2}$$

In terms of string charges (p_i, q_i) we get[2]

$$^{\#}(J \cdot J) = \sum_{1 \le i < j \le n+m} \begin{vmatrix} p_i & p_j \\ q_i & q_j \end{vmatrix} - \sum_{i=1}^{n} d_i^2 \,, \tag{5.3}$$

$$g = \#(P(\mathbf{J}')) + \frac{1}{2} \sum_{i=1}^{n} \{d_i - d_i^2\} = \frac{1}{2}^{\#} (J \cdot J) - \frac{1}{2} \sum_{i=n+1}^{n+m} d_i + 1 \,. \tag{5.4}$$

From the expression for the genus in terms of the charges we see that the presence of 7-branes lift some of the moduli as anticipated in [6]. In case of non BPS junction $\mathbf{J}$ (5.4) gives a lower bound on the genus of J. Equations (5.3) and (5.4) are the main results of this paper. The utility of (5.3) and (5.4) is that only the string prongs at the branes are needed and the explicit realization of the junction is not required.

As an example consider a junction $\mathbf{J}_0$ connecting two D3-branes with the D_4 singularity as shown in fig. 2.

A *necesary* condition for a junction to be BPS is that $(\mathbf{J}, \mathbf{J}) = 2g - 2 + b \ge -1$. BPS junctions with support on the 7-branes representing a D_4 singularity and ending on two D3-branes represent BPS states in the $\mathcal{N} = 2$, $D = 4$ Sp(4) theory with

[2] In case of a single D3-brane, eq. (5.3) written in terms of the Lie algebra (realized on the 7-branes) weight vector $\vec{\lambda}$ and the asymptotic charge (p, q), on the D3-brane, (associated with the junction $\mathbf{J}$) gives $^{\#}(\mathbf{J} \cdot \mathbf{J}) = -\lambda^2 + f(p, q)$ [7]. Here $f(p, q)$ is a quaratic form completely determined by the monodromy K of the 7-brane configuration, $f(p, q) = \frac{1}{2 - TrK} \mathbf{z}^T S K \mathbf{z}$ ($\mathbf{z}^T = (p, q), S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$) [22].

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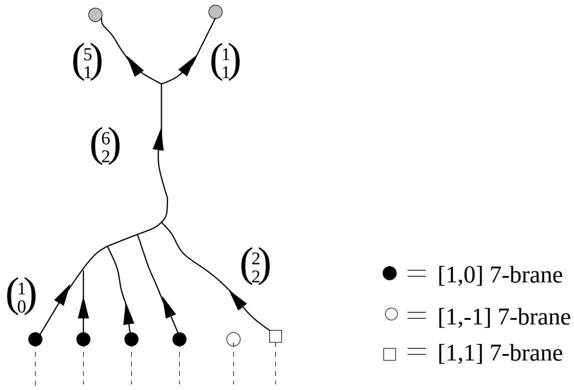

Figure 2: A non BPS junction in the $\mathcal{N} = 2$, $D = 4$ Sp(4) theory with D_4 global symmetry.

four flavors and an antisymmetric tensor. This theory has been studied in [21]. Calculating the self-intersection number of $\mathbf{J}_0$ by adding contributions from the 7-branes and the junction points we get

$$(\mathbf{J}_0, \mathbf{J}_0) = 4(-1) - 2^2 + \begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} = -4\,. \tag{5.5}$$

We can use (5.3) directly to calculate the self-intersection number. Here we have $n = 5$, $m = 2$, $\{(p_i, q_i)|i = 1, \ldots, 7\} = \{(1,0), (1,0), (1,0), (1,0), (2,2), (-1,-1), (-5,-1)\}$ and $\{d_i|i = 1, \ldots, 7\} = \{1,1,1,1,2,1,1\}$. One finds, of course the same result $(\mathbf{J}_0, \mathbf{J}_0) = -4$. Therefore $\mathbf{J}_0$ is not a BPS junction.

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